Mid-term Exam

Time:	10:00–13:00, March 10, 2022.	Course name:	*Algebra II*
Degree:	MMath.	Year:	1st Year, 2nd Semester; 2021–2022.
Course instructor:	Ramdin Mawia.	Total Marks:	30.

Attempt four of the following problems, including problems nº 2 & nº 4.

Group Theory & Field Theory

1. Describe all groups of order 2023 (up to isomorphism). **10**

2. When do we say that a group G is solvable? Show that any group of order p^3q^n is solvable, where $p < q$ are odd primes with $q \neq 1 + p + p^2$ and n is any positive integer. [1] **5**

3. What do we mean by an *extension* of a group G by a group F? When do we say that the extension is *split*? Show that any *split* extension of G by F is a semidirect product of F by G. **10**

4. When do we say that a field extension is *normal*? Let K/k be an algebraic extension. Suppose every $\alpha \in K$ is contained in a normal subextension N (i.e., $k \subset N \subset K$ with N/k normal), then show that K/k is normal. **5**

5. Decide whether the following statements are true or false, with brief but complete justifications (counterexamples, proofs etc.) (**any five**): **10**

   (a) If N is a normal subgroup of G, then G is always isomorphic to $N \times G/N$.

   (b) If N is a normal subgroup of G, then G always has a subgroup isomorphic to G/N.

   (c) If K/k is a field extension of degree 3, then it is a normal extension.

   (d) If $k \subset F \subset K$ is a tower of fields such that F/k, K/F are both algebraic, then K/k is also algebraic.

   (e) If K and L are normal extensions of k, contained in the same extension E of k (i.e., $k \subset K, L \subset E$), then KL is normal over both K and k.

   (f) Let K/k be an extension of degree m and $f(X) \in k[X]$ be an irreducible polynomial of degree coprime to m, then K is not the splitting field of $f(X)$.

   (g) The splitting field of $X^3 + X^2 + X + 1 \in \mathbb{Q}[X]$ is of degree 3 over $\mathbb{Q}$.

   (h) Given a group G, the subgroup $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$ consisting of inner automorphisms, is normal in $\mathrm{Aut}(G)$.

   (i) If $G_1, \ldots, G_n$ are solvable, so is their direct product $G_1 \times \cdots \times G_n$.

   (j) If $1 \to F \to E \to G \to 1$ is a short exact sequence of groups with F and G abelian, then E is necessarily abelian.

[1] It remains solvable even when $q = 1 + p + p^2$, but you don't have to prove this.